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Nonlinear Boundary Value Problems on Semi-Infinite Intervals

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Abstract—A boundary value problem approach to solving nonlinear differential equations on the half line is discussed. Several examples are presented to illustrate the ideas.

1. INTRODUCTION

Nonlinear boundary value problems on $[0, \infty)$ arise in a wide variety of fields. A principal source of such problems is in fluid flow. In boundary layer theory, Blasius-type equations lead to problems on the half-line. However, there are several other fields where such problems arise, e.g., semiconductor circuits [1] and soil mechanics [2]. In addition, singular boundary value problems on finite intervals can be converted into equivalent nonlinear problems on semi-infinite intervals. Over the years, the primary approach to studying such problems has been via an initial value approach and then use of the shooting method.

However, if we knew additional information on the asymptotic behavior of the solution, (e.g., rate of decay or growth, monotonicity properties of the solution and the derivatives) then a boundary value approach can be adopted with considerable success. It is in this spirit that the results in this paper are presented.

In this paper, we present several nonlinear problems which are defined on the semi-infinite interval or can be transformed into one. Our goal is to show that these problems, with apparently significantly different nonlinearities, share several similar features. These features can be utilized in both establishing the existence and uniqueness of solutions and in deriving good numerical approximations.

Nonlinear boundary value problems on the half-line have been studied quite extensively; we give only a few references here [3-6]. In Section 2 of this paper, we discuss how four nonlinear problems on the semi-infinite interval have been studied by conversion to initial value problems. Sections 3, 4, and 5 deal with several semi-infinite interval problems which are studied by a boundary value approach. Section 6 discusses numerical solutions of these problems. In this context, we refer to the work of [7] on parabolic problems. We conclude with some remarks on open questions in Section 7.

2. CONVERSION TO INITIAL VALUE PROBLEMS

In this section, we will present some examples of infinite interval problems and finite interval boundary value problems (possibly singular) which can be converted to initial value problems and then studied numerically using some numerical integration procedures.

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Following Klamkin [8], let us consider the equation of Blasius for the steady two-dimensional flow along a flat plate placed edgewise to the stream. Using similarity transformations, the problem is reduced to solving the infinite interval boundary value problem given by

$$y''' + yy'' = 0, \qquad y(0) = 0, \qquad y'(0) = 0, \qquad y'(\infty) = 2. \tag{2.1}$$

To illustrate the ideas, let us look for a formal series solution $y(x) = \sum c_n x^n$ of (2.1), where the infinity condition is replaced by $y''(0) = \lambda$. It is seen that

$$y(x) = \frac{\lambda x^2}{2!} - \frac{\lambda^2 x^5}{5!} + \frac{11\lambda^3 x^8}{8!} - \cdots . \tag{2.2}$$

Let $z(x)$ be the solution of (2.1) when $\lambda = 1$. Then, it can be seen that

$$y(x) = \lambda^{1/3} z\left(\lambda^{1/3} x\right), \tag{2.3}$$

and further,

$$\lim_{x\to\infty} y'(x) = \lambda^{2/3} \lim_{x\to\infty} z'\left(\lambda^{1/3} x\right) = \lambda^{2/3} \lim_{x\to\infty} z'(x). \tag{2.4}$$

But $\lim_{x\to\infty} y'(x) = 2$, from (2.1). Thus,

$$\lambda = \left\{ \frac{2}{\lim\limits_{x\to\infty} z'(x)} \right\}^{3/2} . \tag{2.5}$$

Therefore, (2.1) can be treated as the system of initial value problems

$$z''' + zz'' = 0, \qquad z(0) = 0, \qquad z'(0) = 0, \qquad z''(0) = 1, \tag{2.6}$$

and

$$y''' + yy'' = 0, \qquad y(0) = 0, \qquad y'(0) = 0, \qquad y''(0) = \left\{ \frac{2}{\lim\limits_{x\to\infty} z'(x)} \right\}^{3/2} . \tag{2.7}$$

Another way of looking at this derivation is to rewrite (2.2) as

$$\lambda^{-1/3} y(x) = \frac{\left(\lambda^{1/3} x\right)^2}{2!} - \frac{\left(\lambda^{1/3} x\right)^5}{5!} + \frac{11\left(\lambda^{1/3} x\right)^8}{8!} - \cdots . \tag{2.8}$$

Now setting

$$\lambda^{-1/3} y(x) = Y \quad \text{and} \quad X = \lambda^{1/3}(x),$$

we have

$$Y(X) = \frac{X^2}{2!} - \frac{X^5}{5!} + \frac{11X^8}{8!} - \cdots . \tag{2.9}$$

In terms of the new variables Y and X, equation (2.1) becomes

$$\begin{gathered} \frac{d^3 Y}{dX^3} + Y \frac{d^2 Y}{dX^2} = 0, \qquad Y(0) = 0, \qquad Y'(0) = 0, \\ Y''(0) = 1 \text{ (coming from } y''(0) = \lambda). \end{gathered} \tag{2.10}$$

And this is precisely the initial value problem (2.6).

Thus, an initial value problem approach to solving (2.1) would be to solve (2.6) and estimate $\frac{dz}{dx}|_{x=\infty}$. Then one would find $\lambda = \{(2)/(\lim_{x\to\infty} z'(x))\}^{3/2}$ and then

$$y(x) = \lambda^{1/3} z\left(\lambda^{1/3} x\right),$$

from (2.3).

This approach to studying problems of type (2.1) has been utilized by several other authors, and one can see the ideas dating back to the work of Lie. We now clarify these ideas by applying them to a couple of other problems, and we will return to the study of these problems later in the paper.

Thus, let us consider the problem of the boundary layer flow of a viscous, electrically conducting incompressible fluid past a semi-infinite plate in the presence of a magnetic field [9]. After suitable transformations, the problem reduces to a coupled system of infinite interval problems

$$\begin{gathered} y''' + yy'' - \beta zz'' = 0, \qquad z'' + \epsilon\left[y z' - zy'\right] = 0, \\ y(0) = y'(0) = 0, \qquad y'(\infty) = 2, \qquad z(0) = 0, \qquad z'(\infty) = 2. \end{gathered} \tag{2.11}$$

Proceeding as in [10], we use the new variables

$$x = A\bar{x}, \qquad y = B\bar{y}, \qquad z = C\bar{z}, \tag{2.12}$$

and then the differential equations in (2.11) become

$$\begin{gathered} \frac{d^3\bar{y}}{d\bar{x}^3} + \frac{\bar{y}\,d^2\bar{y}}{d\bar{x}^2} AB - \beta\frac{AC^2}{B}\frac{\bar{z}\,d^2\bar{z}}{d\bar{x}^2} = 0, \\ \frac{d^2\bar{z}}{d\bar{x}^2} + \epsilon AB\left(\frac{\bar{y}\,d\bar{z}}{d\bar{x}} - \frac{\bar{z}\,d\bar{y}}{d\bar{x}}\right) = 0. \end{gathered} \tag{2.13}$$

Let us now choose A and B, such that

$$AB = 1, \tag{2.14}$$

and set

$$\bar{\beta} = \beta\frac{AC^2}{B} = \beta A^2 C^2. \tag{2.15}$$

Equations (2.13) then reduce to

$$\begin{gathered} \frac{d^3\bar{y}}{d\bar{x}^3} + \bar{y}\frac{d^2\bar{y}}{d\bar{x}^2} - \bar{\beta}\,\bar{z}\frac{d^2\bar{z}}{d\bar{x}^2} = 0, \\ \frac{d^2\bar{z}}{d\bar{x}^2} + \epsilon\left[\bar{y}\frac{d\bar{z}}{d\bar{x}} - \bar{z}\frac{d\bar{y}}{d\bar{x}}\right] = 0. \end{gathered} \tag{2.16}$$

The conditions at 0 reduce to

$$\bar{y}(0) = \left.\frac{d\bar{y}}{d\bar{x}}\right|_{\bar{x}=0} = 0, \qquad \bar{z}(0) = 0. \tag{2.17}$$

If we set the other initial conditions to be

$$\left.\frac{d^2\bar{y}}{d\bar{x}^2}\right|_{\bar{x}=0} = 1, \qquad \left.\frac{d\bar{z}}{d\bar{x}}\right|_{\bar{x}=0} = 1, \tag{2.18}$$

then, by solving (2.16)–(2.18), we find $\bar{y}$ and $\bar{z}$. Also, one notes that the conditions at ∞ on y and z reduce to

$$\left.\frac{B}{A}\frac{d\bar{y}}{d\bar{x}}\right|_{\bar{x}=\infty} = 2, \qquad \left.\frac{C}{A}\frac{d\bar{z}}{d\bar{x}}\right|_{\bar{x}=\infty} = 2,$$

or, using (2.14),

$$A = \left[\frac{1}{2}\,\frac{d\bar{y}}{d\bar{x}}\bigg|_{\bar{x}=\infty}\right]^{1/2}, \qquad \frac{C}{A} = 2\left[\frac{d\bar{z}}{d\bar{x}}\bigg|_{\bar{x}=\infty}\right]^{-1}. \tag{2.19}$$

Note $B^{-1} = A$. We thus find A, B, and C and solve for y and z. Thus, solving the infinite interval problem has again been reduced to solving an initial value problem.

Once again, following [10], if we set

$$x = A^{\alpha_1}\,\bar{x}, \qquad y = A^{\alpha_2}\,\bar{y} \tag{2.20}$$

in (2.1), we get

$$A^{\alpha_2 - 3\alpha_1}\,\frac{d^3\bar{y}}{d\bar{x}^3} + A^{2\alpha_2 - 2\alpha_1}\,\bar{y}\,\frac{d^2\bar{y}}{d\bar{x}^2} = 0.$$

Choose α_1, α_2 such that

$$\alpha_2 - 3\alpha_1 = 2\alpha_2 - 2\alpha_1 \tag{2.21}$$

and (2.1) now reduces to

$$\frac{d^3\bar{y}}{d\bar{x}^3} + \bar{y}\,\frac{d^2\bar{y}}{d\bar{x}^2} = 0. \tag{2.22}$$

The conditions at $\bar{x} = 0$ are

$$\bar{y}(0) = 0, \qquad \bar{y}'(0) = 0.$$

Finally, setting

$$\frac{d^2\bar{y}}{d\bar{x}^2}\bigg|_{\bar{x}=0} = A,$$

we get

$$A^{\alpha_2 - 2\alpha_1}\,\frac{d^2\bar{y}}{d\bar{x}^2} = A,$$

and this leads to

$$\alpha_2 - 2\alpha_1 = 1.$$

Solving this equation together with (2.21), we have

$$\alpha_1 = -\alpha_2 = -\frac{1}{3},$$

and this gives the transformations (from (2.20))

$$x = A^{-1/3}\,\bar{x}, \qquad y = A^{1/3}\,\bar{y}.$$

Finally, the infinity condition reduces to

$$A^{\alpha_2 - \alpha_1}\,\frac{d\bar{y}}{d\bar{x}}\bigg|_{\bar{x}=\infty} = 1$$

or

$$A = \left[\frac{d\bar{y}}{d\bar{x}}\bigg|_{\bar{x}=\infty}\right]^{-3/2},$$

as was seen before.

We consider next the problem of longitudinal impact of nonlinear viscoplastic rods [11]. The corresponding simplified infinite interval boundary value problem is

$$\frac{d^2y}{dx^2} + mx\left(\frac{dy}{dx}\right)^{2-q} = 0, \tag{2.23}$$

with

$$y(0) = 0 \quad \text{and} \quad y(\infty) = 1. \tag{2.24}$$

Proceeding as above, let us set

$$x = A^{\alpha_1}\,\bar{x}, \qquad y = A^{\alpha_2}\,\bar{y},$$

and the requirement that the equation remain invariant gives

$$\frac{\alpha_2}{\alpha_1} = \frac{q+1}{q-1}.$$

Setting the unknown slope $\frac{dy}{dx}|_{x=0}$ to be A, we get

$$\alpha_2 - \alpha_1 = 1, \tag{2.25}$$

so that

$$\alpha_1 = \frac{1}{2}\,(q-1), \qquad \alpha_2 = \frac{1}{2}\,(q+1).$$

The infinity condition gives

$$A^{\alpha_2}\,\bar{y}(\infty) = 1$$

or

$$A = [\bar{y}(\infty)]^{-2/(q+1)}.$$

Thus, we solve the initial value problem

$$\frac{d^2\,\bar{y}}{d\,\bar{x}^2} + m\,\bar{x}\left(\frac{d\,\bar{y}}{d\,\bar{x}}\right)^{2-q} = 0,$$

$$\bar{y}(0) = 0, \qquad \left.\frac{d\,\bar{y}}{d\,\bar{x}}\right|_{\bar{x}=0} = 1,$$

and then find $A = [\bar{y}(\infty)]^{-2/(q+1)}$.

Finally, we have

$$x = A^{\alpha_1}\,\bar{x}, \qquad y = A^{\alpha_2}\,\bar{y}.$$

We conclude this section with a problem that arises in the vibration of a spherical cap [12] and is not an infinite interval problem but a finite interval boundary value problem. The equation is

$$\frac{d^2y}{dx^2} + \frac{x^2}{32y^2} = \frac{\lambda^2}{8}, \tag{2.26}$$

and the associated boundary conditions are

$$y(0) = 0, \qquad 2y'(1) - (1+\nu)\,y(1) = 0. \tag{2.27}$$

The condition $y(0) = 0$ creates the possibility of (2.26) being a singular boundary value problem, and a change of variable $\xi = \frac{1}{x}$ converts (2.26),(2.27) into an infinite interval problem. This approach has been utilized to study both existence and numerical approximations in [13].

Proceeding as above and setting

$$x = A^{\alpha_1}\,\bar{x}, \qquad y = A^{\alpha_2}\,\bar{y}, \tag{2.28}$$

we get

$$\frac{d^2\,\bar{y}}{d\,\bar{x}^2} + \frac{\bar{x}^2}{32\,\bar{y}^2} = \frac{\lambda^*}{8}, \tag{2.29}$$

if we set

$$\alpha_2 - 2\alpha_1 = 2\alpha_1 - 2\alpha_2 \tag{2.30}$$

and

$$\lambda^* = \lambda^2 A^{2\alpha_1 - \alpha_2}. \tag{2.31}$$

The condition at $x = 0$ becomes

$$\bar{y}(0) = 0. \tag{2.32}$$

Setting the missing slope at $x = 0$ to be

$$\left.\frac{dy}{dx}\right|_{x=0} = A,$$

we get

$$\alpha_2 - \alpha_1 = 1 \tag{2.33}$$

and

$$\left.\frac{d\bar{y}}{d\bar{x}}\right|_{\bar{x}=0} = 1. \tag{2.34}$$

Solving (2.30) and (2.33), we get

$$\alpha_1 = 3, \qquad \alpha_2 = 4. \tag{2.35}$$

From the boundary condition at $x = 1$, given by

$$2y'(1) - (1 + \nu)\, y(1) = 0,$$

we get

$$2\,\bar{x}\,\frac{d\bar{y}}{d\bar{x}} = (1 + \nu)\,\bar{y}, \qquad A = \left(\frac{1}{x}\right)^{1/2}.$$

We have thus converted the finite interval boundary value problem to an initial value problem as in the above discussions.

The literature on the conversion of infinite interval boundary value problems to initial value problems is quite extensive; many other references may be seen in [10]. We have chosen the above examples since we will discuss them later in this paper by a different procedure.

3. A BOUNDARY VALUE PROBLEM APPROACH

Consider the nonlinear boundary value problem

$$y'' = 2\sinh y, \qquad y(0) = c, \qquad y(\infty) = 0, \tag{3.1}$$

where $c > 0$.

This interesting problem, which arises in a wide variety of fields viz. colloids, plasmas, semiconductor devices etc., is chosen to illustrate a boundary value approach to infinite interval problems. The approach is easily adaptable to more general problems of the type (3.1) and for details one is referred to [14].

Let us study first an associated finite interval problem

$$\phi'' = 2\sinh\phi, \qquad \phi(0) = c, \qquad \phi(R) = 0, \tag{3.2}$$

where $0 < c$ and $0 < R < \infty$. Clearly, if ϕ_1 and ϕ_2 are solutions of (3.2), we have

$$\phi_1'' - \phi_2'' = 2\left[\sinh\phi_1 - \sinh\phi_2\right].$$

Multiplying by $\phi_1-\phi_2$, integrating from 0 to R and using the boundary conditions, the uniqueness of the solution of (3.2) follows. Finally, the existence of a solution of (3.2) follows by an application of the Leray-Schauder principle, and we refer to [14] for the details. Let us now establish some properties of the solution of (3.2).

PROPERTY A. The solution ϕ of (3.2) satisfies $0 \le \phi(x) \le c$, $\phi'(x) \le 0$ and $\phi''(x) \ge 0$ on $[0, R]$.

PROOF. We first show that $\phi(x) \ge 0$. If $\phi(x)$ can be less than zero at some point in $[0, R]$, then $\phi(x)$ has a negative minimum in $[0, R]$. Let x_0 be the point at which $\phi(x)$ attains this negative minimum. Clearly $x_0 \in (0, R)$ and $\phi'(x_0) = 0$, $\phi''(x_0) \ge 0$. But this is a contradiction to

$$\phi''(x_0) = 2\sinh\phi(x_0) < 0.$$

Thus, $\phi(x) \ge 0$ in $[0, R]$.

It now follows from $\phi'' = 2\sinh\phi$ that $\phi''(x) \ge 0$ on $[0, R]$.

Finally, if $\phi'(x) = k > 0$ at some $x_1 \in (0, R)$, then from $\phi(x_1) \ge 0$ and $\phi'(x_1) > 0$, it follows that there exists an interval $(x_1, x_1 + h) \subset [0, R]$ on which $\phi(x) > 0$. And $\phi''(x) \ge 0$ implies that $\phi'(x)$ is nondecreasing on $[0, R]$. Thus, $\phi'(x) \ge k > 0$ on $[x_1, R]$ and hence, $\phi(x) > 0$ on $[x_1, R]$. This contradicts $\phi(R) = 0$. Hence, $\phi'(x) \le 0$ on $[0, R]$. And since $\phi(0) = c$, it follows that $\phi(x) \le c$ on $[0, R]$.

PROPERTY B. Let $\alpha(x)$ be the solution of

$$\alpha'' = 2\alpha, \qquad \alpha(0) = c, \qquad \alpha(R) = 0. \tag{3.3}$$

Then $\phi(x) \le \alpha(x)$ on $[0, R]$.

PROOF. Let $\delta(x) = \phi(x) - \alpha(x)$. If possible, let $\delta(x) > 0$, for some $x \in [0, R]$. Then, $\delta(x)$ attains a positive maximum in $[0, R]$ at some x_0. Clearly, $x_0 \in (0, R)$. Thus, $\delta(x_0) = \phi(x_0) - \alpha(x_0) > 0$, $\delta'(x_0) = 0$ and $\delta''(x_0) \le 0$. But

$$\begin{aligned}
\delta''(x_0) &= \phi''(x_0) - \alpha''(x_0), \\
&= 2\sinh\phi(x_0) - 2\alpha(x_0) \\
&> 2\sinh\alpha(x_0) - 2\alpha(x_0) \\
&> 0,
\end{aligned}$$

and this is a contradiction.

It must be noted here that we have made use of the fact that

$$\alpha(x) = \frac{c\sinh\sqrt{2}(R-x)}{\sinh\sqrt{2}R} \ge 0, \qquad \text{for } x \in [0, R].$$

Thus, $\delta(x) = \phi(x) - \alpha(x) \le 0$.

PROPERTY C. Let $\beta(x)$ be the solution of

$$\beta'' = 2\,\frac{\sinh c}{c}\,\beta, \qquad \beta(0) = c, \qquad \beta(R) = 0. \tag{3.4}$$

Then, $\beta(x) \le \phi(x)$.

PROOF. Similar to Property B.

PROPERTY D.

$$0 \le \beta(x) \le \phi(x) \le \alpha(x), \qquad x \in [0, R].$$

PROOF. It suffices to show $0 \le \beta(x)$, the rest being Property B and Property C. That $\beta(x) \ge 0$ follows from the fact that the exact solution of (3.4) is

$$\beta(x) = c\,\frac{\sinh a(R-x)}{\sinh aR},$$

where $a = \sqrt{(2\sinh c)/c}$.

Noting further that $\beta(x) > 0$ on $[0, R)$, we obtain the following property.

PROPERTY E. $\phi(x) > 0$ on $[0, R)$, $\phi'(x) < 0$ on $[0, R]$ and $\phi''(x) > 0$ on $[0, R)$.

PROPERTY F. If we denote by ϕ_R the solution to (3.2), where R is the right-hand end point of the x-interval, and if $S > R$, then

$$\phi_S(x) > \phi_R(x) \text{ on } [0, R] \text{ and } \phi'_S(x) > \phi'_R(x) \text{ on } [0, R].$$

PROOF. We have

$$\phi''_R - \phi''_S = 2\left[\sinh\phi_R - \sinh\phi_S\right].$$

Multiplying both sides by $\phi_R - \phi_S$ and integrating from $x = 0$ to $x = X > 0$, we have

$$\begin{aligned}[\phi_R(X) - \phi_S(X)]\,[\phi'_R(X) - \phi'_S(X)] &= \int_0^X \left(\phi'_R - \phi'_S\right)^2 dx \\ &\quad + 2\int_0^X \left(\sinh\phi_R - \sinh\phi_S\right)\left(\phi_R - \phi_S\right) dx > 0.\end{aligned}$$

Setting $X = R$, we have

$$-\phi_S(R)\,[\phi'_R(R) - \phi'_S(R)] > 0.$$

Since $\phi_S(x) > 0$ on $[0, S)$ by Property E, we have

$$\phi'_R(R) - \phi'_S(R) < 0.$$

We can now conclude that $\phi'_R(x) < \phi'_S(x)$ on $[0, R]$ and also $\phi_R(x) < \phi_S(x)$ on $(0, R]$.

Solution of (3.1)

We now consider the sequence of functions $\{\phi_N(x)\}$, where $\phi_N(x)$ is defined on $[0, N]$ and is the solution of (3.2) for $R = N$. Since $\phi'_N(x) < \phi'_{N+m}(x) < 0$ by Properties E and F, we have

$$\left|\phi'_N(x)\right| > \left|\phi'_{N+m}(x)\right| > 0, \qquad x \in [0, N].$$

In particular,

$$\left|\phi'_1(0)\right| > \left|\phi'_2(0)\right| > \cdots > \left|\phi'_{N+m}(0)\right| \geq \left|\phi'_{N+m}(x)\right|,$$

for $x \in [0, N]$, since $\phi'_{N+m}(x)$ is nondecreasing. Thus,

$$\begin{aligned}\left|\phi_{N+m}(x_1) - \phi_{N+m}(x_2)\right| &= \left|\phi'_{N+m}(\xi)\right|\left|x_1 - x_2\right| \\ &\leq \left|\phi'_1(0)\right|\left|x_1 - x_2\right|.\end{aligned}$$

Thus, the family of functions $\{\phi_{N+m}(x)\}$ is equicontinuous on $[0, N]$. Since these functions are also equibounded (note $\left|\phi_R(x)\right| \leq c$, for all R), we can apply Arzela's theorem and conclude that on $[0, N]$ the sequence of functions ϕ_N, $\phi_{N+1}, \ldots$ has a subsequence which converges uniformly (and monotonically by virtue of Property F) to a function $y(x)$ which satisfies

$$y'' = 2\sinh y, \qquad y(0) = c.$$

It is also easy to see that the same limit y would be obtained if we expanded the underlying interval to $[0, N+1]$. Thus, on every finite interval y satisfies

$$y'' = 2\sinh y, \qquad y(0) = c.$$

We now show $y(\infty) = 0$. Since $y(x) \geq 0$, let $\lim_{x\to\infty} y(x) = k > 0$. Let $\epsilon > 0$ be arbitrarily small with $\epsilon < k$. By the uniform convergence result above, it follows that for any preassigned $\epsilon > 0$, there exists $M > 0$ such that

$$\left|y(x) - y_M(x)\right| \leq \epsilon, \qquad \text{for } 0 \leq x \leq M.$$

But $y_M(M) = 0$, and hence, $\left|y(M)\right| \leq \epsilon < k$. Hence, $\lim_{x\to\infty} y(x) = 0$.

The proof of uniqueness of y follows as for (3.2).

Remarks on the Solution of (3.1)

REMARK A. It can be shown that $\lim_{x\to\infty} y'(x) = 0$. If we multiply (3.1) by y' and integrate from $x = 0$ to $x = X$, we obtain

$$[y'(X)]^2 = [y'(0)]^2 + 4\left[\cosh y(X) - \cosh c\right]. \tag{3.5}$$

Since $\lim_{x\to\infty} y(x) = 0$, it follows that

$$\lim_{x\to\infty} [y'(x)]^2 < \infty.$$

If possible, let $\lim_{x\to\infty} y'(x) = k < 0$ and let $\epsilon > 0$ be arbitrarily small, such that $\epsilon + k < 0$. Then there exists $N_1 > 0$, such that

$$\left|y'(x) - k\right| < \epsilon, \qquad \text{for } x \geq N_1.$$

Since $\lim_{x\to\infty} y(x) = 0$, there exists $N_2 > 0$ such that

$$|y(x) - 0| < \epsilon, \qquad \text{for } x \geq N_2.$$

Let $N = \max(N_1, N_2)$. Then, for $x \geq N$,

$$-\epsilon + k < y'(x) < \epsilon + k.$$

Integrating between N and X,

$$(-\epsilon + k)x \mid {}_N^X < y(x) \mid {}_N^X < (\epsilon + k)\, x \mid {}_N^X$$

or

$$(-\epsilon + k)(X - N) + y(N) < y(X) < (\epsilon + k)(X - N) + y(N).$$

Noting that $\epsilon + k < 0$ and $y(N) < \epsilon$, the left and right hand terms eventually become negative and remain so. This implies $y(x) < 0$, which is a contradiction. Thus, $\lim_{x\to\infty} y'(x) \geq 0$. But $y'(x) \leq 0$, and hence,

$$\lim_{x\to\infty} y'(x) = 0.$$

REMARK B. From (3.5), we now conclude that

$$y'(0) = -2\sqrt{\cosh c - 1},$$

and this allows us to compute the solution of (3.1) by treating it as an initial value problem.

REMARK C. Recalling Property B, we note that

$$\alpha'' = 2\alpha, \qquad \alpha(0) = c, \qquad \alpha(R) = 0,$$

and thus,

$$\alpha(x) = c\,\frac{\sinh\sqrt{2}(R - x)}{\sinh\sqrt{2}R}.$$

It can be seen that $\lim_{R\to\infty} \alpha(x) = ce^{-\sqrt{2x}}$ and this function is larger than $\phi(x)$, the solution to (3.1). One could have obtained this upper function by linearizing $2\sinh\phi$ for small ϕ (or equivalently small c). We could also divide both sides by ϕ and apply L'Hospital's rule to arrive at the same differential equation satisfied by α. Similarly, a lower function for $\phi(x)$ on $[0, \infty)$ is given by $\lim_{R\to\infty} \beta(x) = ce^{-\sqrt{[(2\sinh c)/c]}\,x}$.

REMARK D. Finally, we remark that (3.1) is solvable by direct methods and the solution is

$$y(x) = 2\ln\left[\frac{\left(e^{c/2}+1\right)e^{\sqrt{2}x}+\left(e^{c/2}-1\right)}{\left(e^{c/2}+1\right)e^{\sqrt{2}x}-\left(e^{c/2}-1\right)}\right].$$

In fact, multiplying both sides of (3.1) by y' and integrating (note that $y'(x) \to 0$ as $x \to \infty$), we get

$$\frac{dy}{\sqrt{\cosh y - 1}} = \pm 2dx. \tag{3.6}$$

Choosing the negative sign and integrating once again, we get the above expression for y.

It must be noted that choosing the positive sign in (3.6) and continuing to find the solution would yield

$$y(x) = 2\ln\left[\frac{\left(e^{c/2}+1\right)e^{-\sqrt{2}x}+\left(e^{c/2}-1\right)}{\left(e^{c/2}+1\right)e^{-\sqrt{2}x}-\left(e^{c/2}-1\right)}\right].$$

This is not a desirable solution since it has a vertical asymptote at a finite x.

Another model problem where similar behavior occurs is

$$y'' = ay^N, \qquad a > 0, \qquad N > 1, \qquad y(0) = c, \qquad y(\infty) = 0.$$

Note that f' is 0 at $y = 0$, in this case.

4. PROBLEMS INVOLVING LOWER ORDER DERIVATIVES

In Section 3, we discussed how an infinite interval problem can be treated by a boundary value problem approach as opposed to the initial value problem approach of Section 2. However, the model problem did not involve lower order derivatives in y. In this section, we discuss some nonlinear problems where y' is involved and continue with the approach of Section 3.

Thus, let us consider the unsteady, isothermal flow of gas through a semi-infinite porous medium [15] given by the nonlinear partial differential equation

$$\nabla^2\left(p^2\right) = \frac{2\varphi\mu}{k}\frac{\partial p}{\partial t}, \tag{4.1}$$

where $p =$ pressure within the porous medium, $\varphi =$ porosity, $\mu =$ viscosity and $k =$ permeability. Here, it is assumed that the:

(a) flow of the gas follows Darcy's law;
(b) only phase flowing is a gas of constant viscosity;
(c) density of gas is proportional to pressure;
(d) permeability is constant and uniform;
(e) gravitational forces are neglected.

In the special case of unsteady gas flow in a one-dimensional semi-infinite porous medium, equation (4.1) reduces to

$$\left(\frac{\varphi\mu}{k}\right)\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(p\frac{\partial p}{\partial x}\right).$$

Introducing the dimensionless parameter $p = p/(p_0)$ with p_0 being the initial pressure, setting $p_1 = (p_1)/(p_0)$ with p_1 being the pressure at the outflow face, and using the change of variable $p^2 = 1 - \lambda w$, $0 < \lambda < 1$, $\lambda = 1 - p_1^2$, the author in [15] transforms (4.1) into a partial differential equation in w. When one now introduces a new independent variable via a similarity type transformation, we are able to reduce (4.1) to an ordinary differential equation

$$\begin{gathered} z'' + \frac{2x}{\sqrt{1-\lambda z}}z' = 0, \\ z(0) = 1, \qquad z(\infty) = 0, \qquad 0 < \lambda < 1. \end{gathered} \tag{4.2}$$

Equation (4.2) has been studied via an initial value problem approach in [3], in combination with the shooting method. Other related approaches to (4.2) are also referred to in this paper. For other numerical results on (4.2), we refer to [10].

We follow the ideas in Section 3 and study (4.2) by considering a sequence of boundary value problems on $[0, R]$ and let $R \to \infty$. For a detailed analysis, one is referred to [16]; we outline only the key steps here. Thus, the upper and lower solution α and β, as in Section 3, are obtained from the ordinary differential equations

$$\alpha'' + 2x\alpha' = 0 \quad \text{and} \quad \beta'' + \frac{2x}{\sqrt{1-\lambda}}\,\beta' = 0,$$

respectively.

If, in equation (4.2), we make the change of variable $u = 1 - \lambda\, z$, (4.2) is transformed into the nonlinear problem

$$\begin{gathered} u'' + \frac{2x}{\sqrt{u}}\,u' = 0, \\ u(0) = 1-\lambda, \qquad u(\infty) = 1, \qquad 0 < \lambda < 1. \end{gathered} \tag{4.3}$$

A further transformation $u = (1+\beta\, y)^2$, with $1 - \lambda = (1+\beta)^2$ reduces (4.3) to

$$\frac{d}{dx}\,[(1+\beta\, y)\, y'] + 2xy' = 0, \qquad y(0) = 1, \qquad y(\infty) = 0. \tag{4.4}$$

Equation (4.4) arises in phase change problems [17] where the thermal conductivity varies linearly with temperature. When this relationship is not linear, (4.4) generalizes to

$$\frac{d}{dx}\,[(1+\beta(y))y'] + f(x)\, y' = 0. \tag{4.5}$$

Interestingly enough, (4.5) arises in boundary layer problems concerning the effects of thermal radiation on temperature distribution and heat transfer [18]. An initial problem approach to numerical computation of solutions following the ideas of Section 2 may be seen in [10].

However, the ordering of the boundary conditions $y(0)$ and $y(\infty)$ changes the nature and behavior of the solution in these problems considerably, and the analysis requires careful study. For details of the qualitative and numerical results on equations (4.2), (4.4) and (4.5), we refer to [16].

5. A CLASS OF NONLINEAR BOUNDARY VALUE PROBLEMS ON A FINITE INTERVAL

In this section, we illustrate, with the help of two nonlinear boundary value problems on [0,1], how the ideas of Section 3 could be utilized to study singular problems which have been otherwise studied in the literature by conversion to infinite interval problems.

Thus, let us consider the nonlinear problem

$$\begin{gathered} y'' = \frac{-x^2}{32y^2} + \frac{\lambda^2}{8}, \qquad 0 < x \le 1, \\ y(0) = 0, \qquad 2y'(1) - (1+\nu)\, y(1) = 0, \qquad \lambda > 0, \qquad 0 < \nu < 1. \end{gathered} \tag{5.1}$$

This ordinary differential equation arises in the study of the large deflection membrane response of a spherical cap [12]. Earlier results on this problem were in finding numerical solutions [19]. In [13], this problem is converted into an infinite interval problem as follows: first, a change of variable is done by

$$y = x^p\, u, \qquad \frac{1+\nu}{2} < p \le 1,$$

and the problem reduces to

$$x^2 u'' + 2pxu' + p(p-1)u = \frac{-1}{32x^{3p-4}u^2} + \frac{\lambda^2}{8x^{p-2}}.$$

A second change of variable using $x = \frac{1}{t}$, $1 \le t < \infty$ is now performed, and (5.1) now reduces to

$$u'' = \frac{-2(1-p)}{t} u' + \frac{p(1-p)}{t^2} u - \frac{1}{32t^{6-3p}u^2} + \frac{\lambda^2}{8t^{4-p}}.$$

The associated boundary conditions are

$$\left(p - \frac{1+\nu}{2}\right) u(1) - u'(1) = 0, \qquad u(\infty) = 0.$$

Studying this infinite interval problem via a combination of initial value problems and shooting approaches, the author in [13] establishes the following:

(a) there exists at most one positive solution $y(x)$ satisfying $[y(x)/x^p] \to 0$, as $x \to 0^+$ where $\nu < 1$ and $(1+\nu)/2 < p < 1$.

(b) with $\nu < 1$, there exists a positive solution $y(x)$ of (5.1), for which

$$\frac{y(x)}{x} < \frac{1}{2\lambda}, \qquad 0 < x \le 1, \quad \text{and} \quad \lim_{x\to 0} + \frac{y(x)}{x} \le \frac{1}{2\lambda}.$$

It must be noted that Result (a) above leaves open the possibility of multiple solutions satisfying other rates of decay of $y(x)$ to zero as $x \to 0^+$.

Once again, we can use the idea of Section 3 to study equation (5.1). In [13], it was established that there exists a positive solution $y(x)$ satisfying

$$y(x) \ge Cx(\beta - x),$$

for suitable constants C and β. Using this result, we can now show that

$$\alpha(x) = A\frac{x}{2\lambda} - Bx^2,$$

with $A = \sqrt{[(\lambda^2)/(\lambda^2 + (8/\lambda))]}$ and $B = [(A(1-\nu))/(2\lambda(3-\nu))]$ is a positive lower solution of (5.1).

By Result (b) stated above, it follows that

$$\beta(x) = \frac{x}{2\lambda}$$

is a positive upper solution of (5.1).

In view of the boundary condition $y(0) = 0$, we consider first the perturbed problems

$$\begin{aligned} y'' &= -\frac{\lambda^2}{8}\frac{\left([x/2\lambda] + [1/m]\right)^2}{y^2} + \frac{\lambda^2}{8}, \qquad 0 \le x \le 1, \\ y(0) &= \frac{1}{m}, \qquad 2y'(1) - (1+\nu)\,y(1) = 0. \end{aligned} \tag{5.2}$$

That positive upper and lower solutions for (5.2) can be constructed (analogous to those of (5.1)) is now evident. From the well-developed theory [20] of upper and lower solutions for nonlinear boundary value problems, we can now establish the existence of a solution to (5.2). One then shows that the solution of (5.2) is unique. A similar argument shows that a solution of (5.1), if it exists, is unique and no restrictions on the decay of $y(x) \to 0$ as $x \to 0^+$ need to be imposed. Finally, we show that the solution $y_m(x)$ of (5.2) converges to $y(x)$ as $m \to \infty$.

This boundary value approach to (5.1) also allows us to obtain the following qualitative results for (5.1):

(a) let

$$\lambda^3 < 4, \quad B_0 = \frac{1}{32}\left(\lambda^2 - \frac{16}{\lambda} + \sqrt{\lambda^4 + 32\lambda}\right),$$
$$B_1 = \frac{1}{2}\min\left(B_0, \frac{1-\nu}{2\lambda(3-\nu)}\right) \quad \text{and}$$
$$A_1 = 2\lambda\, B_1 + 2\lambda\left(64B_1 + 4\lambda^2\right)^{-1/2}.$$

Then $\beta(x) = A_1\,(x/2\lambda) - B_1\, x^2$ is a positive upper solution of (5.1) and thus, for such λ,

$$\lim_{x\to 0^+} \frac{y(x)}{x} < \frac{1}{2\lambda}$$

answering one of the questions raised in [13] on the maximum radial stress;

(b) recalling that with

$$A = \sqrt{\frac{\lambda^2}{\lambda^2 + (8/\lambda)}}, \qquad B = \frac{A(1-\nu)}{2\lambda(3-\nu)},$$

we have that

$$A\frac{x}{2\lambda} - Bx^2$$

is a positive lower solution of (5.1), and further noting that $A \to 1$ as $\lambda \to \infty$, it follows that

$$\lim_{x\to 0^+} \frac{y}{x} \cong \frac{1}{2\lambda} \quad \text{for large enough } \lambda.$$

We refer to [21] for the technical details of the above results. Finally, we would like to recall that in Section 2 another discussion on conversion of (5.1) to an initial value problem was given.

In the context of flow of non-Newtonian fluids, which are pseudoplastic, the following singular boundary value problem arises:

$$\begin{gathered} y^{1/n} y'' + nx = 0, \qquad 0 < x < 1, \qquad 0 < n < 1, \\ y'(0) = 0, \qquad y(1) = 0. \end{gathered} \tag{5.3}$$

A non-Newtonian fluid is pseudoplastic if the shear stress τ; is related to the strain rate $\frac{\partial u}{\partial y}$ by

$$\tau = \kappa\left(\frac{\partial u}{\partial y}\right)^n, \qquad 0 < n < 1,$$

κ being a positive constant. Equation (5.3) arises from the following third-order infinite interval problem

$$F''' + F(F'')^{2-n} = 0, \qquad F(0) = F'(0) = 0, \qquad F'(\infty) = 1.$$

We refer to [22], where, starting from the boundary layer equations for steady flow over a semi-infinite flat plate, the above equation is derived. If we now use the Crocco-type transformation $u = F'$, $G = F''$, we get

$$G^n\, G'' + (n-1)\, G^{n-1}(G')^2 + u = 0, \qquad G'(0) = 0, \qquad G(1) = 0.$$

Now setting $y = G^n$, we get equation (5.3). For both numerical results using appropriate Runge-Kutta methods and qualitative properties, we refer to [23]. All of these papers utilize an initial value approach. In [24], analytic solutions are attempted with a view to studying qualitative properties, in particular, to study the monotonicity of $y^{1/n}(0)$.

We continue the direct boundary value problem approach to study (5.3). Once again, for the technical details, we refer to [25].

It can be first seen that

$$\beta(x) = 1 - \frac{n}{1+n}\, x^3$$

is a positive upper solution of (5.3), and

$$\alpha(x) = \frac{n}{6}\left(1 - x^3\right)$$

is a positive lower solution of (5.3).

We can then study the sequence of perturbed problems

$$y_m'' + \frac{nx}{y_m} = 0, \qquad y'(0) = 0, \qquad y(1) = \frac{1}{m}. \tag{5.4}$$

By constructing appropriate upper and lower solutions to (5.4) (as we did for (5.3)), we establish existence of a solution to (5.3). We then establish uniqueness of $y_m(x)$. Note that (5.4) is not a singular boundary value problem. Finally, we establish the convergence of $y_m(x)$ to the unique solution $y(x)$ of (5.3).

As in the case of our study of (5.1), we can now obtain qualitative properties of the solution of (5.3) as a consequence of the above results.

We first show that if $y_j(x)$ satisfies

$$\begin{gathered} y_j^{1/j}\, y_j'' + jx = 0, \qquad 0 < x < 1, \qquad 0 < j < 1, \\ y_j'(0) = 0, \qquad y_j(1) = 0, \end{gathered} \tag{5.5}$$

then for $0 < n < m \le 1$,

$$[y_n(x)]^{1/n} \le [y_m(x)]^{1/m} \quad \text{on} \quad (0,1).$$

This shows the monotonicity of $[y_n(x)]^{1/n}$ in n. In [24], the authors obtain, by using initial value approaches, the monotonicity of $[y_n(0)]^{1/n}$.

Further, it can be shown that

$$y_n(0) \to 1 \quad \text{as} \quad n \to \infty$$

and

$$y_n \to 1 \ \text{ uniformly as } \ n \to 0^+ \ \text{ on each subinterval } \ [0,R] \ \text{ of } \ [0,1].$$

6. NUMERICAL SOLUTIONS

Instead of getting into a discussion of the various ways one could study the numerical solution of the semi-infinite interval problems introduced in the earlier sections, we present here one that seems to reflect the qualitative properties and the existence proofs.

We will illustrate the ideas with the help of example (4.2) on the unsteady flow of gas through a semi-infinite porous medium. Recall that the nonlinear problem is

$$\begin{gathered} z'' + \frac{2x}{\sqrt{1-\lambda z}}\, z' = 0, \\ z(0) = 1, \qquad z(\infty) = 0, \qquad 0 < \lambda < 1. \end{gathered} \tag{6.1}$$

It can be easily seen that the solutions $\beta(x)$ and $\alpha(x)$ of the equations

$$\beta'' + \frac{2x}{\sqrt{1-\lambda}}\beta' = 0, \qquad \beta(0) = 1, \qquad \beta(\infty) = 0 \tag{6.2}$$

and

$$\alpha'' + 2x\alpha' = 0, \qquad \alpha(0) = 1, \qquad \alpha(\infty) = 0 \tag{6.3}$$

are given by

$$\beta(x) = 1 - 2(\pi)^{-1/2}(1-\lambda)^{1/4}\int_0^x \exp - \left[(1-\lambda)^{-1/2}t^2\right] dt$$

and

$$\alpha(x) = 1 - 2(\pi)^{-1/2}\int_0^x \exp\left(-t^2\right) dt.$$

Further

$$\beta(x) \le z(x) \le \alpha(x), \tag{6.4}$$

where $z(x)$ is the solution to (6.1).

It can be further seen that if z_R be the solution of

$$z_R'' + \frac{2x}{\sqrt{1-\lambda z_R}} z_R' = 0, \qquad z_R(0) = 1, \qquad z_R(R) = 0, \tag{6.5}$$

and we have similar definitions for α_R and β_R, then $\lim_{R\to\infty}\alpha_R(x) = \alpha(x)$, $\lim_{R\to\infty}\beta_R(x) = \beta(x)$, and $\alpha_R(x) \ge z_R(x) \ge \beta_R(x)$. Hence, given a preassigned $\epsilon > 0$, we find R such that

$$\alpha(x) - \beta(x) < \epsilon, \qquad x \ge R,$$

and then solve the problem

$$y'' + \frac{2x}{\sqrt{1-\lambda y}} y' = 0, \qquad y(0) = 1, \qquad y(R) \in [\beta(R), \alpha(R)]\,. \tag{6.6}$$

Finally, it can be shown that as to the solution of (6.5) or (6.6), if we consider the sequence of linear problems

$$z_{Rn}'' + \frac{2x}{\sqrt{1-\lambda z_{R(n-1)}}} z_{Rn}' = 0, \qquad z_{Rn}(0) = 1, \qquad z_{Rn}(R) = 0,$$

we get a sequence of solutions $\{z_{Rn}\}$ that converges monotonically to z_R.

7. REMARKS

(a) In all the examples discussed in the earlier sections, the solution to the problem and its derivatives have nice monotonicity properties. While this makes the analysis smoother, it does not seem to be essential. A modification of the ideas to include the cases when, for example, the solution is of the form $f(x)\exp(-cx)$ is desirable. It would be interesting to consider the role of weighted normed spaces in this context.

(b) It can be shown in the case of the problem in Section 3 (and in many of the other problems) that one could have studied the truncated problem

$$y'' = 2\sinh\, y, \qquad y(0) = c, \qquad y'(R) = 0, \tag{7.1}$$

and the solution to the semi-infinite problem lies between the solution of (7.1) and the solution of

$$z'' = 2\sinh\, z, \qquad z(0) = c, \qquad z(R) = 0 \tag{7.2}$$

on the interval $[0, R]$. In addition, it can be seen that the solution of

$$\omega'' = 2\sinh \omega, \qquad \omega(0) = c, \qquad \omega(R) + \mu\omega'(R) = 0$$

satisfies

$$z(x) \le \omega(x) \le y(x)$$

on $[0, R]$ for $\mu \in [0, \infty]$. Here, by $\mu = \infty$, we refer to (7.1). Hence, it can be shown that for each $R > 0$ there exists a unique μ, such that the corresponding w and the solution to the semi-infinite interval problem coincide. It would be interesting to estimate this so that the corresponding ω would be the best approximation to the solution of the original problem.

(c) Generalization of the ideas here to systems of second order problems is essential to the study of both third and fourth order problems. An example of a coupled second-third order system was presented in Section 2. In [2], the authors study a nonlinear fourth order semi-infinite interval problem by numerical integration.

(d) In order to obtain more general results for problems of the type

$$y'' = f\left(x, y, y'\right), \qquad y(0) = c, \qquad y(\infty) = 0,$$

it would be desirable to obtain good comparison problems to generate upper and lower solutions on $[0, R]$. In this context, we refer to the results in [3,16].

(e) As remarked in Section 4, the nonlinear problem,

$$\frac{d}{dx}\left[(1+\beta y)\,\frac{dy}{dx}\right] + 2x\,\frac{dy}{dx} = 0, \qquad y(0) = 0, \qquad y(\infty) = 1$$

occurring in phase transitions and the nonlinear problem

$$\frac{d}{dx}\left[\left(1+y^3\right)\frac{dy}{dx}\right] + f(x)\,\frac{dy}{dx} = 0, \qquad y(0) = y_0, \qquad y(\infty) = y_\infty$$

occurring in boundary layers in thermal radiation absorbing and emitting medium differ from the examples studied in the paper in that the corresponding solutions may possess a point of inflection. These problems, however, seem to have most of the other nice features of the examples considered in Sections 2, 3 and 4. Therefore, such problems should be studied in order to unify the theory and numerical solution of nonlinear second order problems on the semi-infinite interval.

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